SERIES WORKSHEET 2 SOLUTION SKETCHES

Note: These are not model solutions, but only sketches/hints towards solutions.

Problem 1. Find the radius of convergence and the interval of convergence (for (6) and (8) just the radius suffices).

(1) $\sum_{n=1}^{\infty} \frac{x^{3n}}{2^n - 3^n}$, (2) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$, (3) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+n-1}$, (4) $\sum_{n=1}^{\infty} \sqrt{n+4^n}x^n$,

(5) $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{n^3}$, (6) $\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n$, (7) $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n}$, (8) $\sum_{n=1}^{\infty} \frac{(n!)^k}{(kn)!}x^n$ $(k \in \mathbb{Z}_{>0})$.

Solution.

(1) $R = \sqrt[3]{3}$, IOC $= (-\sqrt[3]{3}, \sqrt[3]{3})$. Do ratio or root test for R. At the endpoints the general term doesn't go to 0.

(2) $R = 4$, IOC $= [-4, 4]$. Do ratio or root test for R. At the endpoints get convergence by p-series with $p = 4 > 1$.

(3) $R = 1$, IOC $= [1, 3]$. Do ratio or root test for R. At the endpoints get convergence by comparison with p-series with $p = 2 > 1$.

(4) $R = \frac{1}{2}$, IOC $= (-\frac{1}{2}, \frac{1}{2})$. Do ratio or root test for R. At the endpoints the general term doesn't go to 0.

(5) $R = 1$, IOC $= [2, 4]$. Do ratio or root test for R. At the endpoints get convergence by p-series with $p = 3 > 1$.

(6) $R = e$. Do ratio test. (Extra problem: Use Stirling's approximation $\lim_{n\to\infty} \frac{\sqrt{2\pi n}\, n^n e^{-n}}{n!} = 1$ to deduce behaviour at endpoints)

(7) $R = 1$, IOC $= [-1, 1)$. Do ratio or root test for R. At the right endpoint we get the harmonic series, at the left we get the alternating harmonic series (note that n is even iff n^2 is).

(8) $R = k^k$. Do ratio test.

Problem 2. Compute the values of the sums:

(1) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$, (2) $\sum_{n=1}^{\infty} \frac{n}{3^n}$, (3) $\sum_{n=1}^{\infty} \frac{1}{ne^n}$, (4) $\sum_{n=0}^{\infty} \pi^{2n} \frac{(-1)^n}{(2n+1)!}$,

(5) $\sum_{n=1}^{\infty} \frac{(2n)!}{8^n (n!)^2}$ (Hint: Show that $\binom{-\frac{1}{2}}{n} = (-4)^{-n}\frac{(2n)!}{(n!)^2}$), (6) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$.

(7) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

Solution.

(1) e^{-1}. Use the exponential series.

(2) $\frac{3}{4}$. Differentiate the geometric series.

(3) $-\ln(1-e^{-1})$. Use Taylor series for $\ln(1-x)$.

(4) 0. Use Taylor series for $\sin x$.

(5) $\sqrt{2}-1$. Use binomial series using the hint.

(6) $\frac{e+e^{-1}}{2}-1$. Consider the Taylor series for e^x+e^{-x}. (This is a common trick to get rid of the odd terms).

(7) $\frac{\pi}{2\sqrt{3}}$. Use the Taylor series for $\arctan x$.

Problem 3. Express the given functions as power series centered at 0.

(1) $\frac{x^2}{x^4+16}$, (2) $\frac{1+x}{1-x}$, (3) $\sin^2(x)$, (4) $(x+1)e^{x^2}$.

Solution.

(1) $\displaystyle\frac{x^2}{x^4+16} = \frac{x^2}{16}\frac{1}{1-\left(-\frac{x^4}{16}\right)} = \frac{x^2}{16}\sum_{n=0}^{\infty}\left(-\frac{x^4}{16}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{16^{n+1}}x^{4n+2}$.

(2) $\displaystyle\frac{1+x}{1-x} = 1+\frac{2x}{1-x} = 1+2x\sum_{n=0}^{\infty}x^n = 1+\sum_{n=1}^{\infty}2x^n = 1+2x+2x^2+2x^3+\dots$.

(3) $\displaystyle\sin^2(x) = \frac{1}{2}-\frac{1}{2}\cos(2x) = \frac{1}{2}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$.

(4) $\displaystyle(x+1)e^{x^2} = (x+1)\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}+\frac{x^{2n}}{n!} = \sum_{k=0}^{\infty}\frac{x^k}{(\lfloor\frac{k}{2}\rfloor)!}$. (here $\lfloor x\rfloor$ = largest integer that is $\le x$, in particular $\lfloor\frac{2n+1}{2}\rfloor = n = \lfloor\frac{2n}{2}\rfloor$)

Problem 4. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and $\sum_{n=0}^{\infty} a_n$ converges. *Abel's theorem* says that then $\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$. Use this to compute the following sums:

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, (2) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$.

Solution.

(1) $\ln 2$. Use the series for $f(x) = \ln(1+x)$.

(2) $\frac{\pi}{4}$. Use the series for $f(x) = \arctan x$.

Problem 5. The *Bernoulli numbers* B_n are defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Compute B_n for $n = 0, 1, 2, 3, 4, 5, 6$. Show that $\frac{x}{e^x - 1} + \frac{x}{2}$ is even and hence deduce that $B_n = 0$ whenever $n > 1$ is odd.

Solution. Can compute the first terms in the quotient either by long division of power series or by writing $x = (e^x - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$, then multiplying out the product on the right, comparing coefficients, and finally solving recursively for the B_n. This gives:

n	B_n
0	1
1	$-\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$

If a function is even, then its MacLaurin series has only terms with even power (e.g. compare coefficients in the equation $f(x) = f(-x)$). Some algebraic manipulations show that $f(x) = \frac{x}{e^x - 1} + \frac{x}{2}$ is even, hence all its coefficients with an odd x-power are 0. But except for the x^1 coefficient, they all coincide with the coefficients of $\frac{x}{e^x - 1}$.

Remark: On Homework 3 you showed $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. More generally, one can show

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$$

for all integers $k > 0$.

Problem 6. Recall that the Fibonacci numbers are defined recursively by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. We can use power series to derive the explicit formula for F_n as follows. Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$.

(1) Use the recurrence relation and the initial conditions for F_n to deduce $f(x) = \frac{-x}{x^2+x-1}$.

(2) Use partial fraction decomposition to write $f(x)$ as $f(x) = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$ for suitable numbers A, B, α, β.

(3) Use the expression found for f in (2) and the geometric series to deduce

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

by comparing coefficients.

Solution.

(1) $f(x) = \sum_{n=0}^{\infty} F_n x^n = x + \sum_{n=2}^{\infty} F_n x^n = x + \sum_{n=2}^{\infty}(F_{n-1}+F_{n-2})x^n = x + \sum_{n=1}^{\infty} F_n x^{n+1} + \sum_{n=0}^{\infty} F_n x^{n+2} = x + (x+x^2)f(x)$. Now solve for $f(x)$.

(2) $x^2 + x - 1$ has roots α, β with $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$. We then have $\frac{-x}{x^2+x-1} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$ with $A = \frac{-\alpha}{\sqrt{5}}, B = \frac{\alpha}{\sqrt{5}}$.

(3)

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n x^n &= \frac{-x}{x^2+x-1} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} \\
&= \frac{A}{-\alpha}\frac{1}{1-\frac{x}{\alpha}} + \frac{B}{-\beta}\frac{1}{1-\frac{x}{\beta}} \\
&= \frac{A}{-\alpha}\sum_{n=0}^{\infty}\left(\frac{x}{\alpha}\right)^n + \frac{B}{-\beta}\sum_{n=0}^{\infty}\left(\frac{x}{\beta}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{A}{-\alpha}\alpha^{-n} + \frac{B}{-\beta}\beta^{-n}\right)x^n.
\end{aligned}$$

Now compare coefficients to get $F_n = \frac{A}{-\alpha}\alpha^{-n} + \frac{B}{-\beta}\beta^{-n}$. Plug in the values of A, B, α, β found in (2) to get the result (it might be helpful to use the identity $\alpha\beta = -1$).

Problem 7. Use the first order Taylor polynomial and its error bound to show the error bound for the midpoint rule (Hint: First consider one interval $[x_0, x_1]$. Using the Taylor inequality for $|f(x) - T_1(x)|$

show that $\left|\int_{x_0}^{x_1} f(x)\mathrm{d}x - \Delta x f(\overline{x_1})\right| \leq \frac{(\Delta x)^3 M}{24}$ where M is a bound for $|f''|$. Then add up all the error terms for the individual intervals $[x_i, x_{i+1}]$ to get the error bound on $[a, b]$).

Solution. Let the setup be as in the midpoint rule. Let M be such that $|f''(x)| \leq M$ for $x \in [a, b]$. Consider a single interval $[x_i, x_{i+1}]$. Let T_1 denote the first order Taylor polynomial with center $\overline{x_{i+1}} = \frac{x_i + x_{i+1}}{2}$. We have

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x &= \int_{x_i}^{x_{i+1}} f(x) - T_1(x) + T_1(x)\,\mathrm{d}x \\ &= \int_{x_i}^{x_{i+1}} T_1(x)\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x) - T_1(x)\,\mathrm{d}x\end{aligned}$$

Using $T_1(x) = f(\overline{x_{i+1}}) + f'(\overline{x_{i+1}})(x - \overline{x_{i+1}})$ show that $\int_{x_i}^{x_{i+1}} T_1(x)\mathrm{d}x = \Delta x f(\overline{x_1})$. Then we get

$$\begin{aligned}\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x - \Delta x f(\overline{x_{i+1}})\right| &= \left|\int_{x_i}^{x_{i+1}} f(x) - T_1(x)\,\mathrm{d}x\right| \\ &\leq \int_{x_i}^{x_{i+1}} |f(x) - T_1(x)|\,\mathrm{d}x \\ &\leq \int_{x_i}^{x_{i+1}} \frac{M}{2!}\,|x - \overline{x_{i+1}}|^2\,\mathrm{d}x \\ &= \frac{M}{2} \cdot 2\frac{(x - x_{i+1})^3}{3} \\ &= M\frac{(\Delta x)^3}{24}\end{aligned}$$

Now add up all the error bounds:

$$\begin{aligned}\left|\int_a^b f(x)\,\mathrm{d}x - \Delta x\big(f(\overline{x}_1) + \cdots + f(\overline{x}_n)\big)\right| &\leq \left|\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - \Delta x f(\overline{x_1})\right| + \cdots + \left|\int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x - \Delta x f(\overline{x_n})\right| \\ &\leq M\frac{(\Delta x)^3}{24} + \cdots + M\frac{(\Delta x)^3}{24} \\ &= nM\frac{(\Delta x)^3}{24} \\ &= nM\frac{\left(\frac{b-a}{n}\right)^3}{24} \\ &= \frac{M(b-a)^3}{24n^2}\end{aligned}$$

This is the desired error bound.

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